A Trip Through Eigen Land:
Where most roads lead to the direction associated with the largest eigenvalue

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An understanding of eigen theory can provide students with powerful ways of analyzing and understanding systemic-level problems in many areas of mathematics, engineering, and sciences. Most mathematics, engineering, and physics majors will encounter eigen theory at least twice in their undergraduate career: in linear algebra and in differential equations. Research shows that students struggle to bridge their informal and intuitive ways of thinking with the formalization of concepts in linear algebra (Dorier, Robert, Robinet and Rogalski, 2000; Carlson, 1993). While the work presented here is part of a larger study aimed at better understanding this bridge and informing its traversal, our goals in this paper are much more targeted. In this paper we aim to (1) identify important aspects of student thinking about eigen theory that emerged over the course of the teaching experiment, and (2) discuss how these new insights will inform a revision of the hypothetical learning trajectory (Simon, 1995).

Our project team conducted a four-week classroom teaching experiment (Cobb, 2000) during the Fall semester of 2007 in an introductory linear algebra class for university undergraduates, most of whom are majoring in mathematics or engineering. During this time, each class session was videotaped, copies of all student work were retained, and pre- and post- interviews were conducted with individual students. In this short paper we provide a retrospective analysis of some important aspects of student thinking in the eigen unit, and proposed revisions to the instructional sequence that were informed by the insights we have gained through this analysis.

Theoretical Framing

The instructional design of our eigen theory unit draws upon heuristics from two compatible theoretical perspectives: the Models & Modeling Perspective developed by Lesh & Doerr (2003) and the instructional design theory of Realistic Mathematics Education (see e.g. Gravemeijer, 1999). In particular, the overarching instructional design of the unit is guided by a hypothetical learning trajectory that begins with an experientially real starting point and aims to help students build a new mathematical reality in which eigenvectors and eigenvalues come to be meaningful objects in and of themselves. We define a *Hypothetical Learning Trajectory* (HLT) to be a storyline about teaching and learning that occurs over an extended period of time. The storyline includes four aspects, all of which are reflexively related and revisable: (1) Learning goals about student reasoning, (2) A storyline of how students' mathematical learning experience will evolve, (3) The role of the teacher in the storyline, and (4) a sequence of instructional tasks that students will engage in. In our view, a HLT is primarily a tool to be used by a research team interested in studying one or more of the four aspects that constitute a HLT. The creation of the HLT is guided by our intent to help student develop increasingly general and formal ways of thinking about eigen theory.

Important Aspects of Student Thinking: A Retrospective Analysis

A major goal of our teaching experiment was to explore the ways students approach and think about the following basic question about eigen theory: "For what vectors $\mathbf{v}$ and what scalars λ does the equation $B\mathbf{v}=\lambda\mathbf{v}$ have a non-trivial solution?" We would like to clarify that this particular framing of the question is offered here for the benefit of the reader (mathematics education researchers and mathematicians familiar with eigen theory) – obviously notation and language had to be developed with students before such a formally stated version of a "basic question" would be meaningful to them. In the teaching experiment, we approached this basic question by first giving students a real-world Models and Modeling task that was an adaptation of a fairly traditional stochastic matrix problem. This problem, when modeled by iteratative multiplication of the stochastic matrix by an initial state vector yielded a sequence of vectors that converged to the steady state vector of the system. Thus our framing of the basic questions involved a generalization from the notion of steady state vectors to the notion of "same-direction" vectors yielded by matrix multiplication. Our ongoing analysis aims to examine the ways in which our framing of the "basic question" contributed to and constrained students' evolving conceptions of eigenvectors and eigenvalues.

In order to anticipate and make sense of student thinking with the most basic questions about eigen theory, we knew that it would be important to have a sense of the ways in which our students were conceptualizing the mathematical objects and operations relevant to these questions (such as vectors, matrices, and how they interact). What we didn't (and couldn't) know prior to the teaching experiment was the ways in which these ideas would play out during the eigen unit.

In this section of the paper, we will discuss in detail one idea that emerged as a central and powerful way in which students came to reason with and about eigenvectors and eigenvalues. This idea has to do with the relationship between the determinant of a matrix and the dependence relationships among its column vectors. We anticipate that this particular idea and accompanying inscriptions will be central to the RME design heuristic of guided reinvention and emergent models (Gravemeijer, 1999). In particular, we will address the how this idea evolved with regard to classroom activity and discussion, how it developed in the context of the eigen unit, and the ways in which this will inform our revised hypothetical learning trajectory.

In our work with students, determinants were first introduced as a way of measuring the area of the image of the unit square under multiplication by an arbitrary 2x2 matrix. In particular, students were asked to find an expression for the area of the image of the unit square when multiplied by the matrix $\begin{bmatrix} a & b \\ c & c \end{bmatrix}$. (See Figure 1.)

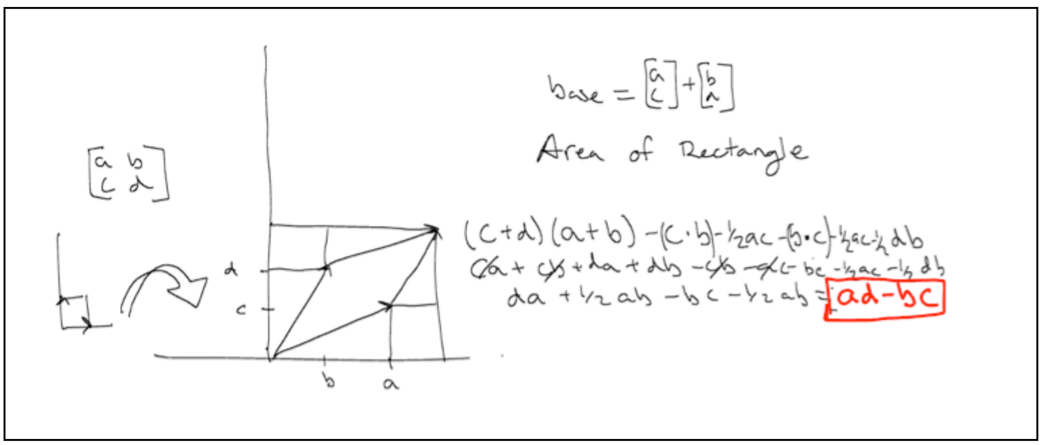

Figure 1. A student shows how he found the area.

After finding this area, students were asked to make predictions about the area of the image of a unit square when multiplied by a 2x2 matrix whose column vectors were linearly dependent. This helped students to develop a visual intuition for thinking about the equivalence of the determinant of a matrix being zero and its column vectors being linearly dependent. Nearly a month after this introduction to determinants, one student, who we will call Karl, explained his thinking about how this idea connects to eigen theory:

When you look at the, uh, vectors, what does the determinant give us? It gives us the area between any two given vectors. And if, if our determinant equals zero, that basically means that the vectors that we're solving for have no area in between. So therefore they lie along the same line.

As he spoke, Karl held his hands in a v-shape, presumably emulating two vectors pointing out from the origin. When he made reference to the determinant being zero, he made a motion of flattening his hands together to indicate that the two vectors now lie along the same line.

This type of reasoning lead us to believe that it might be more intuitive for students to first think about the process of finding eigenvectors and eigenvalues as one whose goal is to find those vectors such that their image lies along the same line as the original vector – and that these vectors could be found by forcing the determinant to be zero.

The "Basic Question" Reframed

As part of our ongoing work to revise and refind our Hypothetical Learning Trajctory for the eigen theory unit, we conclude by offering a reframing of our "basic question" of eigen theory, which we think may better connect to student thinking (see Figure 2):

- When a matrix acts on a vector **v**, the result is a vector B**v**. *Usually,* the resultant vector B**v** does not point in the same direction as the original vector **v**.
- Given a matrix B, is there some vector **w** that, when multiplied by B, results in a vector B**w** that points in the *same* direction as the original vector **w**?

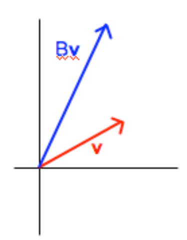

Figure 2: The "Basic Question" Reframed

One thing that we would like to highlight about this reframing is the fact that it is posed as a question that no longer starts with the assumption that input and output vectors point in the same direction – we believe that this may provide more opportunity for students to imagine motions like the ones Karl made with his hands when making his argument about how he could force two vectors to lie along the same line.

We anticipate that if students imagine that these vectors **v** and B**v** lie along the same line, that it must be true that det [**v** B**v**] = 0. Computing the determinant yields an equation that is quadratic in both the first and second components of the vector **v** that, when solved yield the equations of the lines along which the eigenvectors of the matrix B must lie. Such an "eigenvector first" approach has also been documented to be more conceptually accessible to student in Differential Equations (Rasmussen & Blumenfeld, 2007).

For the benefit of the reader, we offer some elaboration of the method described in the previous paragraph: Using $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B\mathbf{v} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$, det [**v** B**v**] = 0, we get

$$\det\begin{bmatrix} x & ax+by \\ y & cx+dy \end{bmatrix} = 0 \quad \Rightarrow x(cx+dy)-(ax+by)y = 0$$

$$\Rightarrow cx^2 + dxy - axy - by^2 = 0$$

$$\Rightarrow cx^2 + (d-a)xy - by^2 = 0 \quad \textbf{(*)}$$

Therefore, if we have a matrix B (so we know the values of *a, b, c,* and *d*) we can use (*) to find the values of x and y.

While it is possible to generalize this method to the 3x3 case by choosing as a third column a nonzero vector that does not lie in an eigenspace, this approach quickly becomes cumbersome as the size of the matrix increases. This provides an ideal opportunity the teacher to introduce the conventional eigenvalue-first approach. Indeed, by this point students will have become somewhat familiar with both eigenvalues and eigenvectors, hence we conjecture that finding either first is sensible since they now have meanings for both.

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